

ON 2-DIMENSIONAL TOPOLOGICAL FIELD THEORIES

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ABSTRACT. In this paper we give a characterization of 2-dimensional topological field theories over a space X as Frobenius bundles with connections over LX , the free loop space of X . This is a generalization of the folk theorem stating that 2-dimensional topological field theories (over a point) are described by finite-dimensional commutative Frobenius algebras. In another direction, this result extends the description of 1-dimensional topological field theories over a space X as vector bundles with connections over X , cf. [5].

In [2], Atiyah introduces the notion of a d -dimensional topological quantum field theory. About the same time, Segal [14] defines the concept of a 2-dimensional conformal field theory, motivated by the problem of avoiding the difficulties of Feynman path-integration in quantum field theory through an axiomatic approach. In [13] he suggests that 2-dimensional conformal field theories based on a manifold X should provide geometric cocycles for some version of elliptic cohomology of X , the home of elliptic genera such as the Witten genus (see [18]). The idea of relating field theories and cohomology theories was elaborated by Stolz-Teichner [16], and their collaboration confirms this relationship in dimension one: the space of 1-dimensional supersymmetric *euclydean* field theories is a classifying space for K-theory (supersymmetry here avoids some topological triviality).

Understanding field theories of various flavors seems like a very interesting problem lying at the intersection of topology, geometry and quantum field theory. One of the main conjectures by Stolz-Teichner states that the space of 2-dimensional supersymmetric *euclydean* field theories is a classifying space for the theory of topological modular forms (see [16] and [15]). In this paper we deal with the simpler case of 2-dimensional *topological* field theories over a space X , of which more is known. A folk theorem states that 2-dimensional topological field theories (over $X = \star$) are given by finite-dimensional commutative Frobenius algebras (see for example [1],[8] or [11]). Topological 1-dimensional field theories over a space X are given by finite-dimensional vector bundles with connections over X , see [5]. In this paper we show that 2-dimensional topological field theories over X are given by Frobenius bundles with connections over LX , the free loop space of X . A Frobenius bundle over LX is a vector bundle over LX whose restriction to X , the space of constant loops in LX , is a bundle of Frobenius algebras,

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and the fiber over an arbitrary loop γ in X , based at a point $x \in X$, admits an action of the Frobenius algebra which is the fiber over the constant loop at x .

A topological field theory is a functor from a bordism category to an algebraic category, usually the category of topological vector spaces. There are various versions of such topological theories, for example one could modify the bordism category and consider *open*, or *open-and-closed* bordisms. Two-dimensional such theories were characterized by Moore-Segal [11] and Lauda-Pfeiffer [9]. One could also replace the target category by the category of complexes (see Costello [4]) or by an arbitrary symmetric monoidal category. Even further, one could replace categories by higher categories and consider *extended* topological field theories. Such theories were characterized in the two-dimensional case by Schommer-Pries [12] and in general by Lurie, who outlines in [10] the proof of the cobordism hypothesis, a conjecture stated by Baez-Dolan [3]. None of these variations on topological field theories will be considered in this paper.

This paper is organized as follows: in section 1 we define the notion of a 2-dimensional topological field theory (2-TFT, for short) over a space X (definition 1.1) in a manner convenient for our purposes, and the notion of a Frobenius bundle with connection over LX (definition 1.3). We then state the main theorem 1.2 which establishes the equivalence of the two notions. Section 2 is dedicated to the proof of the theorem. Further consequences of our definition of Frobenius bundles such as Frobenius actions and $\text{Diff}(S^1)^+$ -actions are relegated to section 3. We also talk about holonomy along closed surfaces and rank-one 2-TFTs which are basically S^1 -bundle gerbes with connections.

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1. DEFINITIONS AND STATEMENT OF RESULTS

Field theories. A d -dimensional field theory over a space X is generally defined to be a smooth functor of symmetric monoidal categories

$$E : \mathcal{B}^d(X) \rightarrow \mathbf{Vect}.$$

The objects in the category $\mathcal{B}^d(X)$ are pairs (Y, γ) consisting of a closed oriented $d - 1$ manifold Y and a smooth map $\gamma : Y \rightarrow X$. The morphisms between two such objects (Y_1, γ_1) and (Y_2, γ_2) are pairs (Z, Σ) , with Z an oriented d -manifold such that $\partial Z = Y_1 \amalg \bar{Y}_2$, where \bar{Y} denotes Y with opposite orientation, and $\partial \Sigma = \gamma_1 \amalg \gamma_2$. The category \mathbf{Vect} is the category of topological vector spaces over a field k , which is usually taken to be \mathbf{R} or

C. “Monoidal” means disjoint union \amalg and tensor product \otimes in the category $\mathcal{B}^d(X)$ and \mathbf{Vect} , respectively. The functor E is compatible with these monoidal structures, that is

$$E(\gamma_1 \amalg \gamma_2) = E(\gamma_1) \otimes E(\gamma_2),$$

and similarly on morphisms. Moreover, E takes the empty set to our ground field k . “Functoriality” means that glueing of bordisms in $\mathcal{B}^d(X)$ corresponds to composition of linear maps in \mathbf{Vect} . A functor is *smooth* if it maps a smooth family (i.e. parametrized by smooth manifolds) of objects in the source category to a smooth family of objects in the target category, and similarly for morphisms.

There are various flavors of field theories according to the geometric structures we require on bordisms: *topological* (no structure), *euclidean* (flat Riemannian metric), *conformal* (conformal structure) etc. The easiest example of a topological field theory (TFT, for short) is for $d = 1$, in which case it entails to a parametrization-invariant parallel transport associated to a vector bundle over X , which in turn is just a vector bundle with connection over X (see [5]).

We will modify slightly the definition above to avoid technical difficulties arising from glueing bordisms in X along common boundaries. One way to deal with this is to consider objects along with collars, and glue along collars. Our approach is to replace the composition of bordisms by decomposition. The price to pay is to give up the beautiful categorical language or modify the definition of category accordingly. In order to avoid set-theoretical issues, we should also require that all the vector spaces appearing in the definition below are subspaces of a *fixed* infinite-dimensional topological vector space, let us say k^∞ . Since we are dealing with *topological* field theories, it turns out that all the vector spaces appearing below are finite dimensional.

Definition 1.1. A *2-dimensional topological field theory* E over X assigns smoothly to a union of loops $\amalg \gamma : \amalg S^1 \rightarrow X$ in X a topological vector space $E(\amalg \gamma) = \otimes E(\gamma)$ and to a surface in X , i.e. a map Σ from an oriented surface Z in X a continuous linear map $E(\Sigma) : E(\partial \Sigma_{in}) \rightarrow E(\partial \Sigma_{out})$, where the boundary $\partial Z = \partial Z_{in} \amalg \partial Z_{out}$ splits into incoming and outgoing boundary according to whether the orientation of the circles coincides with the induced orientation of the surface or its reverse, so that the properties below hold:

- (1) (monoidal structure preserving) As noted above, if $\amalg \gamma$ is a union of loops in X , we require

$$E(\amalg \gamma) = \otimes E(\gamma).$$

Moreover, we should ask that $E(\emptyset) = \mathbf{C}$. Also, if Σ_1 and Σ_2 are two bordisms in X , then we should have

$$E(\Sigma_1 \amalg \Sigma_2) = E(\Sigma_1) \otimes E(\Sigma_2).$$

- (2) (compatibility under decomposition) If $\Sigma : Z \rightarrow X$ is decomposed along a (union of) circle(s) Y_0 in Z so that $Z = Z_1 \amalg_{Y_0} Z_2$, and

$\Sigma_i := \Sigma|_{Z_i}$, $i = 1, 2$, we have

$$E(\Sigma) = E(\Sigma_2) \circ E(\Sigma_1)$$

where the left-hand side is a map $E(\partial\Sigma_{in}) \rightarrow E(\partial\Sigma_{out})$, and the right-hand side is a composition $E(\partial\Sigma_{in}) \rightarrow E(\Sigma|_{Y_0}) \rightarrow E(\partial\Sigma_{out})$.

- (3) (invariance under diffeomorphisms)

$$E(\Sigma \circ \Phi) = E(\Sigma),$$

for Φ an arbitrary diffeomorphism of surfaces that is the identity on the boundary.

- (4) (identity preserving) Let Σ_γ be the “constant” bordism over γ , i.e. the cylinder over γ (since our theory is topological, the height of the cylinder is irrelevant here). Then

$$E(\Sigma_\gamma) = 1_{E(\gamma)}.$$

The purpose of this article is to understand 2-dimensional topological field theories over a manifold X . Let LX denote the free loop space of the manifold X . Our main result is the following

Theorem 1.2. *There is a 1-1 correspondence:*

$$\left\{ \begin{array}{c} \text{2-dim topological field theories} \\ \text{over } X \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Frobenius bundles with connections} \\ \text{over } LX \end{array} \right\}$$

A Frobenius bundle with connection on a loop space LX encodes some algebraic data (multiplication and comultiplication maps coming from 8-like loops in X) and some geometric data (parallel transport along paths in LX -or cylinders in X) in a compatible manner. More precisely, we have the following

Definition 1.3. A *Frobenius bundle with connection* on LX , the free loop space on a manifold X , is a vector bundle A over LX together with the following data:

- For a loop γ in X , denote by A_γ the fiber of the bundle A at γ . If $\gamma = \gamma_1 * \gamma_2$ is the concatenation of γ_1 and γ_2 , then there are maps:

$$\mu : A_{\gamma_1} \otimes A_{\gamma_2} \rightarrow A_\gamma, \text{ and } \nu : A_\gamma \rightarrow A_{\gamma_1} \otimes A_{\gamma_2}$$

called multiplication (fusion), respectively co-multiplication (fission).

- Each point $x \in X$ determines a constant loop γ_x at x . There are unit and co-unit maps

$$\eta : k \rightarrow A_{\gamma_x}, \text{ and } \varepsilon : A_{\gamma_x} \rightarrow k.$$

The counit maps give rise to the nondegeneracy condition: $\varepsilon\mu$ is nondegenerate at each constant loop γ_x in X .

- A *connection* on the bundle A over LX , which assigns smoothly to any path $\Gamma : I = [0, 1] \rightarrow LX$ a linear map $A_{\Gamma_0} \rightarrow A_{\Gamma_1}$. This assignment maps a constant path to the identity, is compatible under

decomposition of paths and satisfies the following strong invariance property: two paths in LX that describe the same surface in X give rise to the same parallel transport (two paths $\Gamma, \Gamma' : I \rightarrow LX$ describe the same surface if their adjoint maps $\tilde{\Gamma}, \tilde{\Gamma}' : I \times S^1 \rightarrow X$ are obtained one from the other by precomposition with a diffeomorphism of $I \times S^1$; compare with the definition of a superficial connection in [17]).

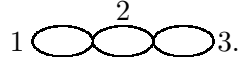
These data are subject to the following conditions:

- (1) (compatibility of fusion/fission with parallel transport) If $(1, 2)$ and $(1', 2')$ are two pairs of concatenated loops and Γ is a path from 1 to $1'$ and Γ' is a path from 2 to $2'$ so that for any $t \in I$, $(\Gamma(t), \Gamma'(t))$ is a pair of concatenated loops, then the following diagrams commute

$$\begin{array}{ccc} 1 \otimes 2 & \xrightarrow{\mu_{12}} & 12 \\ P(\Gamma) \otimes P(\Gamma') \downarrow & & \downarrow P(\Gamma \star \Gamma') \\ 1' \otimes 2' & \xrightarrow{\mu_{1'2'}} & 1'2' \end{array} \quad \begin{array}{ccc} 12 & \xrightarrow{\nu_{12}} & 1 \otimes 2 \\ P(\Gamma \star \Gamma') \downarrow & & \downarrow P(\Gamma) \otimes P(\Gamma') \\ 1'2' & \xrightarrow{\nu_{1'2'}} & 1' \otimes 2' \end{array}$$

(We simplify notation and write i instead of A_i .)

- (2) (associativity) Consider the following 3-petal loop (in X)



Then the diagram below commutes

$$\begin{array}{ccc} 1 \otimes 2 \otimes 3 & \xrightarrow{\mu_{12} \otimes 3} & 12 \otimes 3 \\ 1 \otimes \mu_{23} \downarrow & & \downarrow \mu_{(12)3} \\ 1 \otimes 23 & \xrightarrow{\mu_{1(23)}} & 123. \end{array}$$

- (3) (co-associativity) Referring to the picture of the 3-petal loop above, the following diagram commutes

$$\begin{array}{ccc} 123 & \xrightarrow{\nu_{(12)3}} & 12 \otimes 3 \\ \nu_{1(23)} \downarrow & & \downarrow \nu_{12} \otimes 3 \\ 1 \otimes 23 & \xrightarrow{1 \otimes \nu_{23}} & 1 \otimes 2 \otimes 3. \end{array}$$

- (4) (compatibility of fusion and fission) The following diagram commutes

$$\begin{array}{ccc} 12 \otimes 3 & \xrightarrow{\mu_{(12)3}} & 123 \\ \nu_{1(23)} \downarrow & & \downarrow \nu_{1(23)} \\ 1 \otimes 2 \otimes 3 & \xrightarrow{\mu_{1(23)}} & 1 \otimes 23 \end{array}$$

The diagram with the arrows reversed and μ 's and ν 's interchanged also commutes.

- (5) (compatibility of (co)units with parallel transport) For points x, y in X , and $\gamma : I \rightarrow X$ a path in X connecting x and y , the following diagrams commute

$$\begin{array}{ccc}
 & A_{\gamma_x} & \\
 \eta_x \nearrow & & \searrow P(x; \gamma\bar{\gamma}) \\
 k & & A_{\gamma\bar{\gamma}} \\
 \eta_y \searrow & & \nearrow P(y; \gamma\bar{\gamma}) \\
 & A_{\gamma_y} &
 \end{array}
 \qquad
 \begin{array}{ccc}
 & A_{\gamma_x} & \\
 P(\gamma\bar{\gamma}; x) \nearrow & & \searrow \varepsilon_x \\
 A_{\gamma\bar{\gamma}} & & k \\
 P(\gamma\bar{\gamma}; y) \searrow & & \nearrow \varepsilon_y \\
 & A_{\gamma_y} &
 \end{array}$$

where $P(x; \gamma\bar{\gamma})$ and $P(y; \gamma\bar{\gamma})$ denote parallel transport from the loop at x , respectively at y , to the loop $\gamma\bar{\gamma}$. Similarly, for $P(\gamma\bar{\gamma}; x)$ and $P(\gamma\bar{\gamma}; y)$.

- (6) (compatibility of units and fusion with parallel transport) Let 1 denote a loop in X based at $x \in X$, and let λ_x denote the constant loop at x . The following diagram is commutative

$$\begin{array}{ccc}
 & 1 \otimes \lambda_x & \\
 \eta_x \nearrow & & \searrow \mu_{1x} \\
 1 & \xrightarrow{P(1; 1\lambda_x)} & 1\lambda_x
 \end{array}$$

- (7) (compatibility of counits and fission with parallel transport) With the notation of (6), the following diagram commutes

$$\begin{array}{ccc}
 & 1 \otimes \lambda_x & \\
 \nu_{1x} \nearrow & & \searrow \varepsilon_x \\
 1\lambda_x & \xrightarrow{P(1\lambda_x; 1)} & 1
 \end{array}$$

An easy consequence of the theorem above is the following

Theorem 1.4. *There is a 1-1 correspondence:*

$$\left\{ \begin{array}{c} \text{2-dim homotopical field theories} \\ \text{over } X \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Frobenius bundles with flat} \\ \text{connections over } LX \end{array} \right\}$$

Definition 1.5. A 2-dimensional field theory E is *homotopical* if the definition 1.1 holds with the condition (3) of invariance under diffeomorphisms replaced by

- (3') (invariance under homotopies)

$$E(\Sigma') = E(\Sigma),$$

whenever Σ and Σ' are smoothly homotopic.

A Frobenius bundle with a *flat* connection over LX is a Frobenius bundle over LX with connection so that the parallel transport is invariant under homotopies of paths in LX . Note that this is a stronger notion than the previous one, and it implies it. In particular, the invariance of parallel transport along paths in LX that describe the same surface is automatically implied.

Remarks. 1. The data of a Frobenius bundle with connection over LX expresses the information contained in a 2-TFT over a space X in a generators-and-relations type theorem.

2. A Frobenius bundle over LX , when restricted to a constant loop at a point $x \in X$, encodes the information of a commutative Frobenius algebra. Thus, a Frobenius bundle over LX , when restricted to X , the space of constant loops in LX , gives rise to a bundle of commutative Frobenius algebras. We will see later (subsection 3.1) that the fiber A_γ over an arbitrary loop γ in X based at $x \in X$ admits an action of the Frobenius algebra A_x , the fiber over the constant loop at x .

3. Property (5) in the definition above will allow us to define a field theory for surfaces in X with no incoming or outgoing boundary, in particular the holonomy along closed surfaces in X . In fact, property (5) is equivalent to the following more general property

(5') For points x, y in X , and $\Gamma : D^2 \rightarrow X$ a disk in X containing x and y , the following diagrams commute

$$\begin{array}{ccc}
 & A_{\gamma_x} & \\
 \eta_x \nearrow & & \searrow P(x;\gamma) \\
 k & & A_\gamma \\
 \eta_y \searrow & & \nearrow P(y;\gamma) \\
 & A_{\gamma_y} &
 \end{array}
 \qquad
 \begin{array}{ccc}
 & A_{\gamma_x} & \\
 P(x;\gamma) \nearrow & & \searrow \varepsilon_x \\
 A_\gamma & & k, \\
 P(y;\gamma) \searrow & & \nearrow \varepsilon_y \\
 & A_{\gamma_y} &
 \end{array}$$

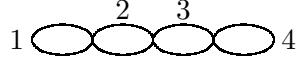
where γ is the restriction of Γ to the boundary $S^1 = \partial D^2$, $P(x;\gamma)$ and $P(y;\gamma)$ denote parallel transport along Γ from the loop at x , respectively loop at y , to the loop γ .

This property immediately implies property (5) and it is obtained from (5) by applying further a parallel transport along paths in the loopspace.

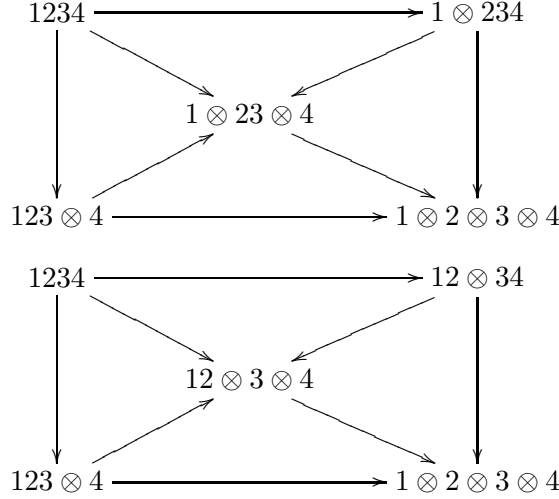
4. One can modify the definition of a 2-dimensional topological field theory over X by requiring that the theory associates a vector space to any loop in X up to reparametrization, i.e. to any *string* in X , and to any surface up to diffeomorphism (not necessarily identity on the boundary) in X it associates a linear map between the fibers corresponding to the boundary. In this situation we obtain a similar description of 2-TFTs over X as in

theorem 1.2, where we replace the definition of a Frobenius bundle with connection over LX by a simpler one in which we drop conditions (6) and (7) in the definition 1.3, and the picture which gives rise to associativity and co-associativity of (2) and (3) is replaced by an “honest” 3-petal loop, i.e. the three loops share a common point.

Higher (co)associativity. The compatibility of parallel transport with fusion/fission and the (co)associativity in the definition above implies the higher (co)associativity. Consider, for example, the following like-loop (in X):



There are three ways to break the big loop going around the loops 1, 2, 3 and 4 into the four little loops using the fission maps and the coassociativity of the fission maps. Let us show that this is independent of the possible choices. For that, let us look at the following diagrams:



where the maps are the obvious fission maps combined possibly with canonical reparametrizations of the loops. The inside-the-square diagrams are commutative making the outer squares commutative. This shows independence on the various choices to “arrive” from 1234 to the little loops labelled 1 through 4. One can proceed now by induction to show the higher co-associativity. In a similar manner, one deals with the higher associativity of the fusion product. Commutativity of fusion and fission implies the commutativity of the above higher fusion and fission.

2. PROOF OF THE THEOREM

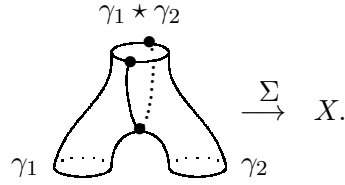
“ \rightarrow ” Let us first show how a 2-TFT E over X gives rise to a Frobenius bundle with connection over LX . For a loop γ in X , define $A_\gamma = E(\gamma)$. If

we consider the smooth family of loops in X parametrized by the “universal loop space” LX , the field theory provides a family of vector spaces $\{A_\gamma\}$ parametrized by LX , i.e. a vector bundle A over LX . We should endow the bundle A with a multiplication and a comultiplication map and a connection.

(Co)multiplication. Let $\gamma = \gamma_1 \star \gamma_2$ be an 8-like loop in X . Define the comultiplication map

$$\nu : A_\gamma \rightarrow A_{\gamma_1} \otimes A_{\gamma_2}, \quad \nu := E(\Sigma),$$

where Σ is the map from the pair of pants into X such that at the top end restricts to γ and at the bottom two ends restricts to $\gamma_1 \amalg \gamma_2$ (see picture).



Similarly, define the multiplication map

$$\mu : A_{\gamma_1} \otimes A_{\gamma_2} \rightarrow A_\gamma, \quad \mu := E(\bar{\Sigma})$$

where $\bar{\Sigma}$ is the map Σ from the pair of pants with reversed orientation (read down-up) into X .

(Co)units. Let $x \in X$ be an arbitrary point and $\Sigma_x : D^2 \rightarrow X$ be the constant map with value x , viewed as a cobordism in X from the empty set to the constant loop at x . Define the unit

$$\eta_x : k \rightarrow A_x, \quad \eta_x := E(\Sigma_x).$$

Similarly, define the co-unit

$$\varepsilon_x : A_x \rightarrow k, \quad \varepsilon_x := E(\bar{\Sigma}_x).$$

where $\bar{\Sigma}_x : \bar{D}^2 \rightarrow X$ is the map Σ_x viewed as a cobordism in X from the constant loop at x to the empty set.

Connection. Let now $\Gamma : I \rightarrow LX$ be a path in the loop space between γ_1 and γ_2 . We can interpret Γ as a map $\Sigma : I \times S^1 \rightarrow X$, i.e. as a bordism between γ_1 and γ_2 . Define

$$P(\Gamma) : A_{\gamma_1} \rightarrow A_{\gamma_2}, \quad P(\Gamma) := E(\Sigma).$$

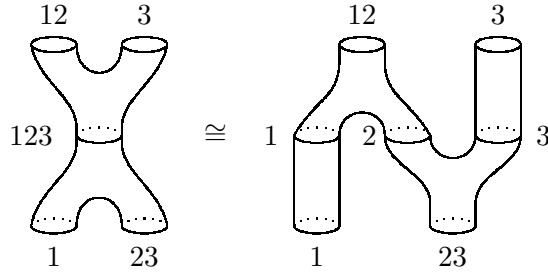
The map P defined on the path space of LX satisfies the following properties:

- (1) $P(\Gamma_\gamma) = 1_{A_\gamma}$, where Γ_γ is a constant path at $\gamma \in LX$.
- (2) (Invariance under diffeomorphisms) $P(\Gamma') = P(\Gamma)$, where $\Gamma' : I \rightarrow LX$ is the path in LX , whose adjoint map $\check{\Gamma}' : I \times S^1 \rightarrow X$ is given by $\check{\Gamma}' = \check{\Gamma} \circ \Phi$, where $\Phi : I \times S^1 \rightarrow I \times S^1$ is an arbitrary diffeomorphism which is the identity on the boundary.

- (3) (Compatibility under decomposition) $P(\Gamma) = P(\Gamma_2) \circ P(\Gamma_1)$, if Γ decomposes as $\Gamma = \Gamma_2 \star \Gamma_1$.

These are exactly the properties that define a connection (see the definition above) on the bundle A over LX , i.e. parallel transport along paths in LX , or along cylinders in X .

The properties (1)-(3) are easy to see. Property (4) expressing the compatibility of fusion and fission is a consequence of the following diffeomorphism of surfaces (read up-down and down-up):

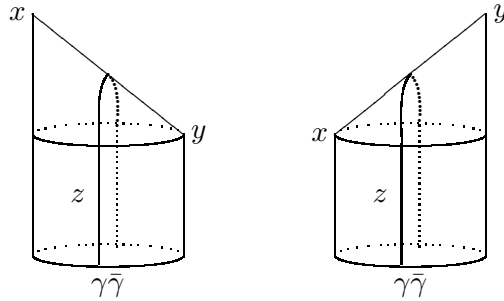


These surfaces map in the indicated way to the 3-petal loops.

Property (5) is a consequence of the following two observations. First, if $\Sigma : Z \rightarrow X$ is a surface in X that is constant on a neighborhood about a point $z \in Z$, and Z' is obtained from Z by collapsing the neighborhood to a point and Σ' is the induced map, then $E(\Sigma) = E(\Sigma')$. This happens because the field theory is smooth, and Σ' is a limit (as $t \rightarrow 0$) of maps Σ_t related by diffeomorphisms to Σ , so

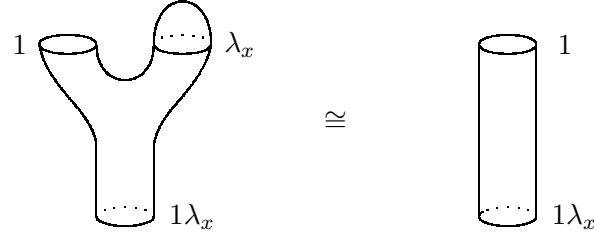
$$E(\Sigma_t) = E(\Sigma) \longrightarrow E(\Sigma'), \text{ as } t \longrightarrow 0.$$

Second, we notice that the bordisms in X in the picture below are diffeomorphic



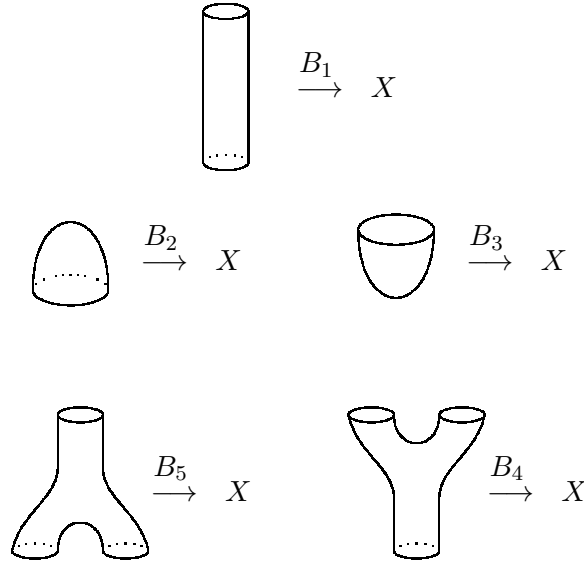
The two bordisms in X drawn above are constant along vertical planes perpendicular to the plane of the paper; for example, along the curves labeled by z , the maps are constant equal to z , where z is a point on the path γ joining x and y . The two bordisms (read up-down) give rise to the two compositions $k \rightarrow A_{\gamma\bar{\gamma}}$ appearing in (5). Read down-up, they give rise to the two compositions $A_{\gamma\bar{\gamma}} \rightarrow k$.

Properties (6) and (7) are easy to see, reflecting the diffeomorphism in the picture below (read up-down and down-up) in the case of a Frobenius algebra (or a 2-TFT over a point)



This ends one direction in the proof of the theorem.

“←” Conversely, start with a Frobenius bundle A with connection over LX . We would like to produce a 2-TFT over X . To each loop γ in X we associate the vector space $E(\gamma) := A_\gamma$. The field theory is determined by specifying the linear maps corresponding to bordisms between loops in X . Each such bordism in X can be recovered (via gluing) from the following “basic” bordisms in X (read up-down):



We shall specify the functor E on such bordisms, and then show that, for an arbitrary bordism, E is independent of the various decompositions of the bordism into basic bordisms. Let P denote the parallel transport map along paths in the loop space LX determined by the connection on the bundle A over LX .

B_1 -type bordism: let γ_1, γ_2 denote the top loop, respectively the bottom loop, of the bordism B_1 . Define

$$E(B_1) : A_{\gamma_1} \rightarrow A_{\gamma_2}, \quad E(B_1) := P(\Gamma),$$

where Γ is the path in the loop space determined by the cylinder, connecting γ_1 and γ_2 .

B_2 -type bordism: let γ denote the bottom loop of B_2 , and let a point on the surface (that does not lie on the boundary) mapping to $x \in X$. Define

$$E(B_2) : k \rightarrow A_\gamma, \quad E(B_2) := P(\Gamma_x) \circ \eta_x,$$

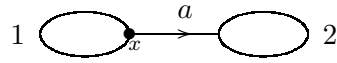
where $\eta_x : k \rightarrow A_x$ is the unit of the Frobenius algebra A_x and Γ_x is the map from the surface into X , viewed as a path in the loop space LX from the constant path at x to the loop γ . This is independent of the various choices since the unit structure maps are compatible with the parallel transport, by the property (5) in the definition of a Frobenius bundle with connection, or its equivalent (5').

B_3 -type bordism: let γ denote the top loop of B_3 , and let a point on the surface as before that maps to $x \in X$. Define

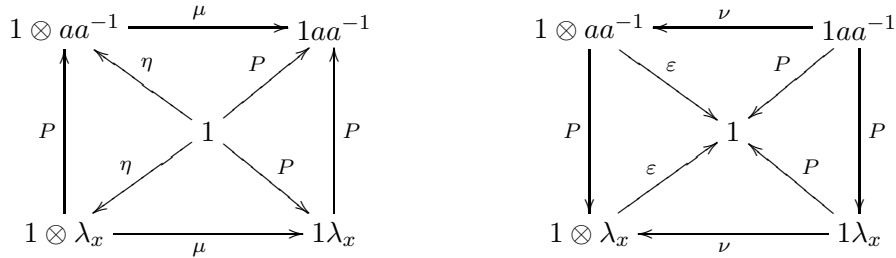
$$E(B_3) : A_\gamma \rightarrow k, \quad E(B_3) := \varepsilon_x \circ P(\Gamma_x),$$

where $\varepsilon_x : A_x \rightarrow k$ is the counit of the Frobenius algebra A_x and Γ_x is the map from the surface into X , viewed as a path in LX from the loop γ to the constant loop at x . This is independent of the various choices since the counit structure maps are compatible with the parallel transport by property (5').

Before we proceed to define the field theory for the pairs of pants of the type B_4 and B_5 we describe a special type of interaction between two loops in X . Specifically, consider the following picture



consisting of two loops in X and a path between the loops, labeled respectively 1, 2 and a . The basepoint of the loop 1 maps to the point $x \in X$. Denote by λ_x the constant loop at x . In the diagrams below



we would like to say that the upper triangles commute, i.e. $\mu\eta = P$, respectively $\varepsilon\nu = P$. This is true for the lower triangles, by the compatibility of units and fusion, respectively counits and fission with parallel transport. The left-hand side triangles commute since (co)units are compatible with parallel transport. The right-hand side triangles also commute since parallel transport is compatible with gluing of paths. The outside squares commute since (co)multiplication is compatible with parallel transport. This gives the required commutativity.

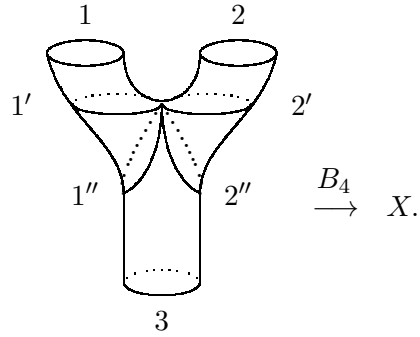
From this we obtain the following diagrams

$$\begin{array}{ccccc}
 1 \otimes 2 & \xrightarrow{P \otimes 2} & 1aa^{-1} \otimes 2 \\
 \downarrow 1 \otimes P & \searrow \eta & \nearrow \mu \otimes 2 & \downarrow \mu \\
 & 1 \otimes aa^{-1} \otimes 2 & \\
 \nearrow 1 \otimes \mu & \downarrow 1 \otimes P & \nearrow \nu \otimes 2 & \downarrow \nu \\
 1 \otimes aa^{-1}2 & \xrightarrow{\mu} & 1aa^{-1}2
 \end{array}$$

$$\begin{array}{ccccc}
 1 \otimes 2 & \xleftarrow{P \otimes 2} & 1aa^{-1} \otimes 2 \\
 \downarrow 1 \otimes P & \swarrow \varepsilon & \nwarrow \nu \otimes 2 & \downarrow \nu \\
 & 1 \otimes aa^{-1} \otimes 2 & \\
 \nearrow 1 \otimes \nu & \downarrow 1 \otimes P & \nearrow \nu & \downarrow \nu \\
 1 \otimes aa^{-1}2 & \xleftarrow{\nu} & 1aa^{-1}2
 \end{array}$$

with all the inside n-gons commutative, making the outer square commutative. The commutativity of these diagrams will allow us to say that the field theory is well defined for pairs of pants mapping into the space X .

B_4 -type bordism:



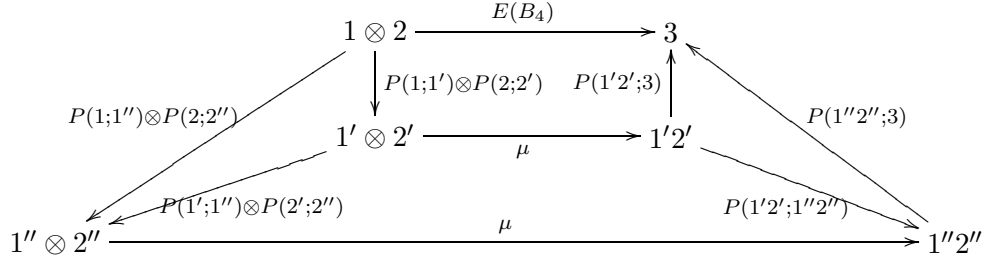
Let γ_1 and γ_2 denote the top two loops and γ_3 the bottom loop. Let $\gamma_{1'}$ and $\gamma_{2'}$ be the loops corresponding to the circles labelled $1'$ and $2'$ in the picture.

Define $E(B_4) : A_1 \otimes A_2 \rightarrow A_3$ by

$$E(B_4) := P(\Sigma_{(1'2')3}) \circ \mu_{1'2'} \circ P(\Sigma_{11'}) \otimes P(\Sigma_{22'}),$$

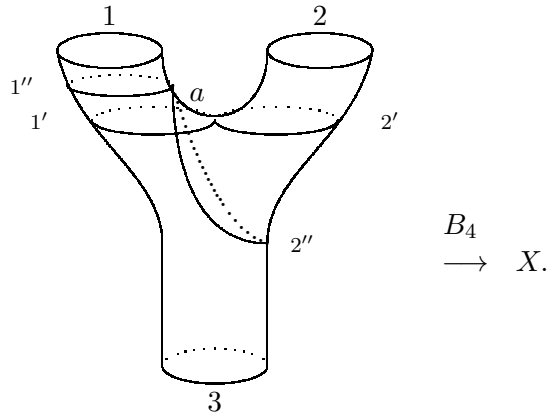
where $\Sigma_{(1'2')3}$ is the bordism in X between $\gamma_{1'} \star \gamma_{2'}$ and γ_3 , $\mu_{1'2'}$ is the multiplication map determined by $\gamma_{1'} \star \gamma_{2'}$, $\Sigma_{11'}$ is the bordism in X between γ_1 and $\gamma_{1'}$ and $\Sigma_{22'}$ is the bordism in X between γ_2 and $\gamma_{2'}$.

The definition is independent on the choice of the intermediate 8-like loop. Indeed, let us consider first the case when the intermediate 8-like loops share a common point as the loops $1'2'$ and $1''2''$ in the picture above (we shall simplify the notation by writing for example simply 1 for the fiber A_1 over the loop γ_1 and $P(11')$ for the parallel transport along the bordism $\Sigma_{11'}$ etc., when no possibility of confusion arises). In the diagram below



all the inside diagrams are commuting expressing either the compatibility of the parallel transport under gluing of paths (of loops) or the compatibility of fusion and parallel transport. This shows that $E(B_4)$ is well defined in this case.

In the situation when the two 8-like loops do not share their junction point (see the picture below), we proceed as follows.

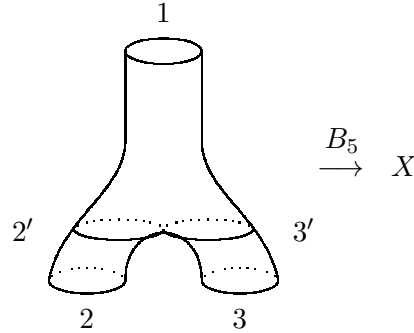


Let a denote the path (in X) between the junction points of the two 8-like loops. Again, in the diagram below

$$\begin{array}{ccccc}
 1 \otimes 2 & \xrightarrow{P \otimes P} & 1'' \otimes a^{-1}a2' & \xrightarrow{P} & 1'' \otimes 2'' \\
 P \otimes P \downarrow & & \downarrow \mu & & \downarrow \mu \\
 1''aa^{-1} \otimes 2' & \xrightarrow{\mu} & 1''aa^{-1}2' & \xrightarrow{P} & 1''2'' \\
 P \downarrow & & \downarrow P & & \downarrow P \\
 1' \otimes 2' & \xrightarrow{\mu} & 1'2' & \xrightarrow{P} & 3
 \end{array}$$

all the inside squares commute. Indeed, the upper left square commutes by the considerations made before defining the field theory for bordisms of type B_4 . The upper right and the lower left square commute by the compatibility of fusion and parallel transport. Finally, the lower right square commutes since parallel transport is compatible under glueing of paths. Reading off the commutativity of the outer square, we obtain the desired independence.

B_5 -type bordism:



Let γ_1 be the top loop and γ_2 and γ_3 be the bottom two loops. Let $\gamma_{2'}$ and $\gamma_{3'}$ be the loops corresponding to the circles labelled $2'$ and $3'$ in the picture. Define $E(B_5) : A_1 \rightarrow A_2 \otimes A_3$ by

$$E(B_5) = P(2'; 2) \otimes P(3'; 3) \circ \nu_{2'3'} \circ P(1; 2'3'),$$

where $P(1; 2'3')$ is the parallel transport along the path in LX between γ_1 and $\gamma_{2'} \star \gamma_{3'}$ determined by (the restriction of) B_4 by adjunction, $\nu_{2'3'}$ is the comultiplication map determined by $\gamma_{2'} \star \gamma_{3'}$, $P(2'; 2)$ is the parallel transport along the path determined by the restriction of B_4 joining $\gamma_{2'}$ and γ_2 and $P(3'; 3)$ denotes the parallel transport along the path between $\gamma_{3'}$ and γ_3 . As for B_4 -type bordisms, the definition is independent on the choice of the intermediate 8-like loop. The key ingredient we use here is property (7) in the definition of a Frobenius bundle, expressing the compatibility of counits and fission with parallel transport.

For an *arbitrary* bordism in X , i.e. a map $\Sigma : Z \rightarrow X$ from a compact oriented smooth surface Z (possibly with boundary) into X , we decompose Σ into basic bordisms B_1 - B_5 and for each basic bordism we apply the previous construction. The theory that we obtain is certainly compatible under decomposition of bordisms, by the very construction. The only thing left to check is that our theory is *topological*.

Invariance under diffeomorphisms. The theory E we constructed is topological if, for any bordism $\Sigma : Z \rightarrow X$ in X and any diffeomorphism of surfaces $\Phi : Z' \rightarrow Z$ that is the identity on the boundary, we have

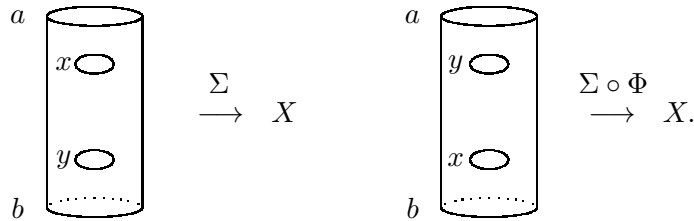
$$E(\Sigma \circ \Phi) = E(\Sigma).$$

This is equivalent to saying that the field theory E is well-defined on Σ , i.e. it is independent on how we decompose Σ . If Σ is one of the basic bordisms B_1 - B_5 , we have already seen that this is the case. For an arbitrary bordism Σ , this reduces to

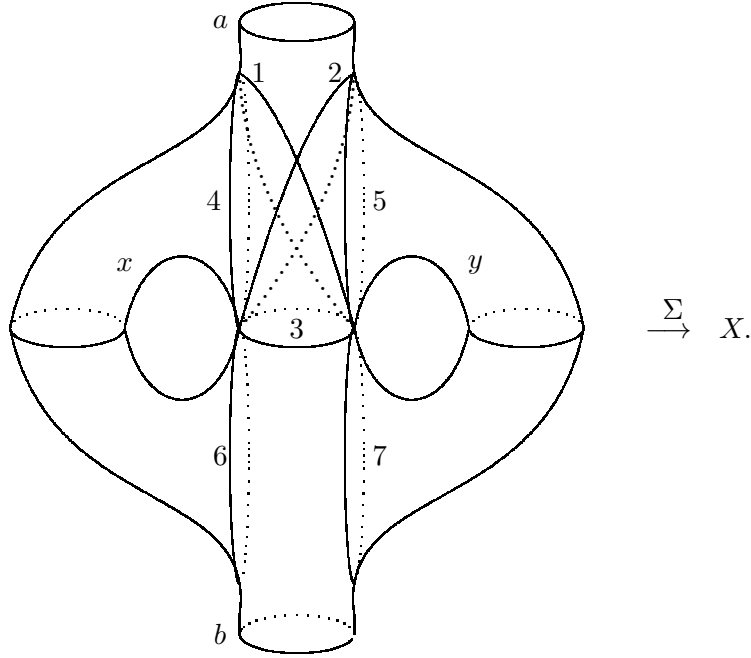
- the associativity of fusion and co-associativity of fission expressed by properties (2) and (3) in the definition 1.3 of a Frobenius bundle
- the higher associativity and co-associativity of section 1
- compatibility of fusion and fission contained in property (4).

Let us illustrate this in a couple of examples, which expose the main properties used.

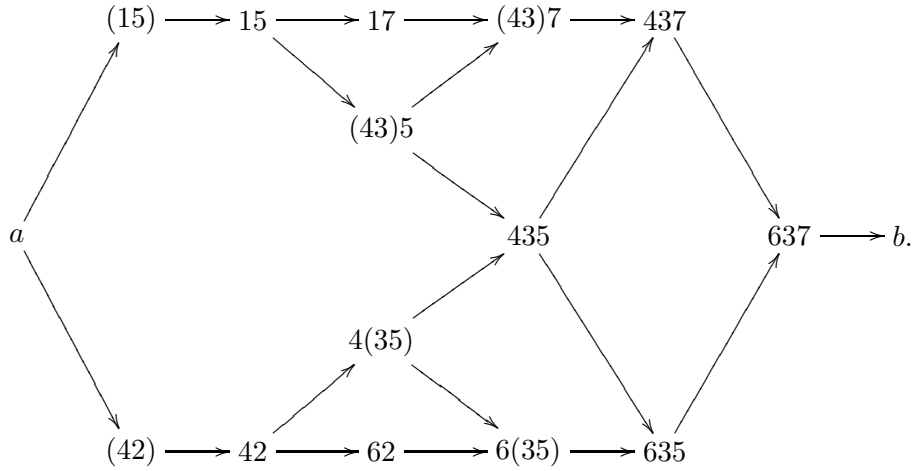
Consider first a genus two surface Z with one incoming boundary component labelled a and one outgoing boundary component labelled b , mapping in two different ways into X , as in the picture



Here Φ is a diffeomorphism of the surface Z that interchanges the holes labelled x and y and is constant on the boundary. Each such bordism can be encoded as two different ways in which the bordism Σ can be decomposed, as indicated by the picture

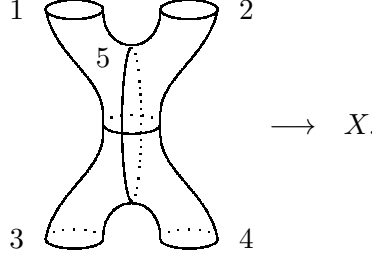


The two different ways to travel from a to b along the bordism Σ appear in the following diagram as the upper path respectively lower path.

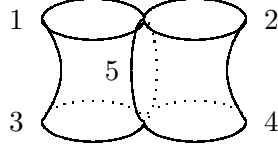


All the small n -gons in the diagram are commutative, expressing the compatibility of fusion and fission with parallel transport, or the co-associativity property (3), as in the two ways to reach from a to 435, or the associativity property (2) appearing in the map $637 \rightarrow b$. We conclude that the outer diagram is commutative, the top path giving us the linear map $E(\Sigma \circ \Phi)$ according to our rules to split a bordism into basic bordisms, and the bottom path giving us $E(\Sigma)$.

The second example we would like to consider is of a genus zero surface in X with two incoming boundary components labelled 1 and 2 and two outgoing boundary components labelled 3 and 4.



There are basically two ways to descend from top to bottom according to our rules of parallel transport along cylinders and fusion and fission maps. One possible way is to fuse the loops 1 and 2, use parallel transport and then disperse into the loops 3 and 4. Another possible way is to split loop 1 into the loops 3 and the loop labelled 5 and then fuse the loops 2 and 5 to reach the loop 4. To show that the result does not depend on the possible ways boils down to considering the following picture in X



and read off the following diagram

$$\begin{array}{ccccc}
 1 \otimes 2 & \xrightarrow{\mu_{12}} & 12 & & \\
 P \downarrow & & P \downarrow & \searrow P & \\
 35 \otimes 2 & \xrightarrow{\mu_{(35)2}} & (35)2 = 3(52) & \xrightarrow{P} & 34 \\
 \nu_{35} \downarrow & & \nu_{3(52)} \downarrow & & \downarrow \nu_{34} \\
 3 \otimes 5 \otimes 2 & \xrightarrow{\mu_{52}} & 3 \otimes 52 & \xrightarrow{P} & 3 \otimes 4.
 \end{array}$$

All the small n -gons are commutative making the exterior n -gon commutative. The key property we use here is the compatibility of fusion and fission, i.e. property (4) in the definition 1.3.

Thus, we have also constructed a topological field theory E over X from a Frobenius bundle with connection over LX . The two constructions (2-TFTs \rightsquigarrow Frobenius bundles, Frobenius bundles \rightsquigarrow 2-TFTs) are clearly inverses of each other. This concludes our proof of the theorem.

3. FURTHER REMARKS

3.1. Frobenius actions. Let γ be a loop in X based at $x \in X$. We will show that A_γ admits an A_x action and coaction. Let us define first a map

$\mu : A_x \otimes A_\gamma \rightarrow A_\gamma$, as the composition

$$A_x \otimes A_\gamma \xrightarrow{\mu_{x\gamma}} A_{x\gamma} \xrightarrow{P(x\gamma;\gamma)} A_\gamma,$$

where $P(x\gamma;\gamma)$ denotes the obvious parallel transport between the loop $x\gamma$ and the loop γ . To see that this map defines an action, we have to check that the diagram

$$\begin{array}{ccc} A_x \otimes A_x \otimes A_\gamma & \xrightarrow{1 \otimes \mu} & A_x \otimes A_\gamma \\ \mu_x \otimes 1 \downarrow & & \downarrow 1 \otimes \mu \\ A_x \otimes A_\gamma & \xrightarrow{\mu} & A_\gamma \end{array}$$

is commutative. It suffices to notice that in the diagram below all but the front face are commutative

$$\begin{array}{ccccc} & & A_x \otimes A_{x\gamma} & & \\ & \nearrow 1 \otimes \mu_{x\gamma} & \downarrow & \nwarrow 1 \otimes P(x\gamma;\gamma) & \\ A_x \otimes A_x \otimes A_\gamma & \xrightarrow{1 \otimes \mu} & A_x \otimes A_\gamma & & \\ \downarrow \mu_x \otimes 1 & & \downarrow \mu_{xx\gamma} & & \downarrow \mu_{x\gamma} \\ A_x \otimes A_\gamma & \nearrow \mu_{xx\gamma} & A_{xx\gamma} & \nwarrow P(xx\gamma;x\gamma) & A_{x\gamma} \\ & \xrightarrow{\mu_{x\gamma}} & & & \end{array}$$

making the front face commute as well. Similarly, we define a coaction map $\nu : A_\gamma \rightarrow A_x \otimes A_\gamma$ as the composition

$$A_\gamma \xrightarrow{P(\gamma;x\gamma)} A_{x\gamma} \xrightarrow{\nu_{x\gamma}} A_x \otimes A_\gamma,$$

where $P(\gamma;x\gamma)$ denotes the obvious parallel transport between the loop γ and the loop $x\gamma$. To see that this map indeed defines a co-action, we have to check that the diagram

$$\begin{array}{ccc} A_\gamma & \xrightarrow{\nu} & A_x \otimes A_\gamma \\ \nu \downarrow & & \downarrow 1 \otimes \nu \\ A_x \otimes A_\gamma & \xrightarrow{\nu_x \otimes 1} & A_x \otimes A_x \otimes A_\gamma \end{array}$$

commutes. This is done as before. Next, we will check that the two actions are compatible in the sense that the following diagram commutes

$$\begin{array}{ccc}
 A_x \otimes A_\gamma & \xrightarrow{\mu} & A_\gamma \\
 1 \otimes \nu \downarrow & & \downarrow \nu \\
 A_x \otimes A_x \otimes A_\gamma & \xrightarrow{\mu_x \otimes 1} & A_x \otimes A_\gamma.
 \end{array}$$

First, let us notice that

$$\begin{aligned}
 \nu \mu &= \nu_{x\gamma} \circ P(\gamma; x\gamma) \circ P(x\gamma; \gamma) \circ \mu_{x\gamma} \\
 &= \nu_{x\gamma} \circ \mu_{x\gamma}.
 \end{aligned}$$

Then, consider the following diagram

$$\begin{array}{ccccc}
 A_x \otimes A_\gamma & \xrightarrow{\mu_{x\gamma}} & & & A_{x\gamma} \\
 \downarrow 1 \otimes \nu & \searrow 1 \otimes P(\gamma; x\gamma) & & \nearrow P(x\gamma; x\gamma) & \downarrow \nu_{x\gamma} \\
 & A_x \otimes A_{x\gamma} & \xrightarrow{\mu_{xx\gamma}} & A_{xx\gamma} & \\
 & \nearrow 1 \otimes \nu_{x\gamma} & & \downarrow \nu_{xx\gamma} & \\
 & & A_{xx} \otimes A_\gamma & \xrightarrow{P(xx; x) \otimes 1} & A_x \otimes A_\gamma \\
 & \nearrow \mu_x \otimes 1 & & & \\
 A_x \otimes A_x \otimes A_\gamma & \xrightarrow{\mu_x \otimes 1} & & & A_x \otimes A_\gamma.
 \end{array}$$

All the inside n-gons are commutative (note that $A_{xx} = A_x$ and $P(xx; x)$ is the identity on A_x), and therefore the outer square is commutative. This proves the compatibility of module and comodule structures. We conclude that the fibers admit a Frobenius action of the Frobenius algebra over their basepoints, justifying hopefully our terminology of “Frobenius bundles”.

Lemma 3.1. *(Frobenius actions are compatible with reparametrizations of loops.) Let $\gamma : S^1 \rightarrow X$ be a loop in X based at $x \in X$, and let the loop $\tilde{\gamma} = \gamma \circ \varphi$ based at $y \in X$, for some diffeomorphism φ of S^1 preserving its orientation. The following diagrams commute*

$$\begin{array}{ccc}
 A_x \otimes A_\gamma & \xrightarrow{\mu} & A_\gamma \\
 P(x; y) \otimes P(\gamma; \tilde{\gamma}) \downarrow & & \downarrow P(\gamma; \tilde{\gamma}) \\
 A_y \otimes A_{\tilde{\gamma}} & \xrightarrow{\mu} & A_{\tilde{\gamma}}
 \end{array}
 \qquad
 \begin{array}{ccc}
 A_\gamma & \xrightarrow{\nu} & A_x \otimes A_\gamma \\
 P(\gamma; \tilde{\gamma}) \downarrow & & \downarrow P(x; y) \otimes P(\gamma; \tilde{\gamma}) \\
 A_{\tilde{\gamma}} & \xrightarrow{\nu} & A_y \otimes A_{\tilde{\gamma}}
 \end{array}$$

where $P(\gamma; \tilde{\gamma})$ is the parallel transport along the canonical path Γ between γ and $\tilde{\gamma}$ and $P(x; y)$ is the parallel transport along the paths of constant loops between the loop at x and the loop at y , determined by the path Γ .

Proof. The proof consists in following the definitions. Let us verify that the first diagram commutes (the second diagram is dealt with in a similar fashion). Indeed, consider the diagram

$$\begin{array}{ccccc}
 A_x \otimes A_\gamma & \xrightarrow{\mu} & A_\gamma & & \\
 \downarrow P(x;y) \otimes P(\gamma;\tilde{\gamma}) & \searrow \mu_{x\gamma} & \nearrow P(x\gamma;\gamma) & & \downarrow P(\gamma;\tilde{\gamma}) \\
 & A_{x\gamma} & & & \\
 & \downarrow P(x\gamma;y\tilde{\gamma}) & & & \\
 & A_{y\tilde{\gamma}} & & & \\
 \nearrow \mu_{y\tilde{\gamma}} & & \searrow P(y\tilde{\gamma};\tilde{\gamma}) & & \\
 A_y \otimes A_{\tilde{\gamma}} & \xrightarrow{\mu} & A_{\tilde{\gamma}} & &
 \end{array}$$

All the inside diagrams are commutative, making the outer diagram commutative. \square

3.2. On holonomy. Start with a Frobenius bundle A with connection over LX . Let Z be a closed surface and $\Sigma : Z \rightarrow X$ be a closed “surface” in X . Let a and b two distinct points on Z and x , respectively y , their images via Σ . The *holonomy* around the closed surface Σ is defined to be the composition

$$k \xrightarrow{\eta_x} A_x \xrightarrow{E(\tilde{\Sigma})} A_y \xrightarrow{\varepsilon_y} k,$$

where A_x, A_y denote the constant loops at the points x and y , and $\tilde{\Sigma}$ denotes the surface Σ in X viewed as a bordism between the constant loop at x to the constant loop at y . To see that holonomy is well-defined, i.e. it does not depend on our choices, let us remark that it is enough to consider the case of a genus zero surface with one boundary component mapping into X and show that the transport along such surfaces is well-defined. These cases are exactly covered by the B_2 and B_3 -type bordisms in X that appeared in the proof of the theorem, for which we showed independence on the various choices. This makes holonomy well-defined.

3.3. $\text{Diff}(S^1)^+$ -action on a Frobenius bundle. Let A be a Frobenius bundle with connection over LX . We define an $\text{Diff}(S^1)^+$ -action on the bundle A as follows. Let γ be a loop in X and φ an element of $\text{Diff}(S^1)^+$. Define a map

$$R_\varphi : A_\gamma \rightarrow A_{\gamma \circ \varphi}, \quad R_\varphi := P(\gamma; \gamma \circ \varphi),$$

where $P(\gamma; \gamma \circ \varphi)$ is the parallel transport along the *canonical* path $\Gamma : I \rightarrow LX$ in the loopspace LX between γ and $\gamma \circ \varphi$ given by

$$\Gamma_t = \gamma \circ ((1-t)\text{id}_{S^1} + t\varphi).$$

Here we identify a diffeomorphism of S^1 with a diffeomorphism of $I = [0, 1]$ preserving the endpoints. If we let the loop γ vary in LX , for each diffeomorphism φ of S^1 , we get a bundle map (still denoted) $R_\varphi : A \rightarrow A$.

Lemma 3.2. *The maps R_φ , $\varphi \in \text{Diff}(S^1)^+$ define a right $\text{Diff}(S^1)^+$ -action on the Frobenius bundle A over LX . In other words,*

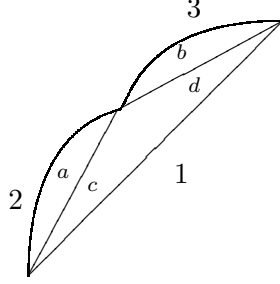
$$R_\psi \circ R_\varphi = R_{\varphi \circ \psi},$$

for φ, ψ in $\text{Diff}(S^1)^+$.

Proof. We have to check that for any loop γ in X and any diffeomorphisms φ, ψ in $\text{Diff}(S^1)^+$, we have

$$P(\gamma; \gamma\varphi\psi) = P(\gamma\varphi; \gamma\varphi\psi) \circ P(\gamma; \gamma\varphi).$$

If $\psi \geq \varphi \geq \text{id}_{S^1}$, then this is indeed the case, since the path from id_{S^1} to $\varphi\psi$ passes through the intermediate step φ . Otherwise we encounter a situation like in the picture below



where 1 labels the identity on S^1 , 2 labels the diffeomorphism φ and 3 denotes the composition $\varphi\psi$. Then 2 is the concatenation of the paths labelled a and d , and 3 is the concatenation of the paths labelled c and b . We would like to show that

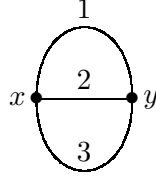
$$P(1; 3) = P(2; 3) \circ P(1; 2),$$

where $P(i; j) := P(\gamma \circ i; \gamma \circ j)$. We have

$$\begin{aligned} P(2; 3) \circ P(1; 2) &= P(cd; ad) \circ P(cb; cd) \circ P(cd; cb) \circ P(1; cd) \\ &= P(cd; ad) \circ P(1; cd) \\ &= P(1; 3). \end{aligned}$$

All the other displacements of the diffeomorphisms 1, 2 and 3 are obtained through a repetitive process of the situations described above. This proves the lemma. \square

3.4. Rank-one 2-dimensional topological field theories. A rank-one (i.e. the fibers are one-dimensional) TFT over X gives rise to a fusion bundle with superficial connection over LX in the sense of Waldorf. Indeed, we only need to check a couple of things. First, is to give a fusion map that is strictly associative. To see this, consider the paths (constant at the endpoints) labelled 1, 2, and 3 between two points x and y in X as in the picture



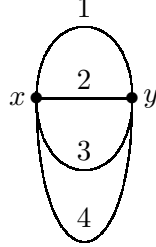
Each such picture should provide a *fusion* map (in the sense of Waldorf)

$$\lambda_{123} : A_{1\bar{2}} \otimes A_{2\bar{3}} \rightarrow A_{1\bar{3}},$$

where the bar notation is used for traveling along a path backwards. We define this to be the composition

$$\lambda_{123} := P(1\bar{2}2\bar{3}; 1\bar{3}) \circ \mu_{(1\bar{2})(2\bar{3})}.$$

We would have to check the following associativity (refer to the picture below)



$$\begin{array}{ccc} 1\bar{2} \otimes 2\bar{3} \otimes 3\bar{4} & \xrightarrow{\lambda_{234}} & 1\bar{2} \otimes 2\bar{4} \\ \lambda_{123} \downarrow & & \downarrow \lambda_{124} \\ 1\bar{3} \otimes 3\bar{4} & \xrightarrow{\lambda_{134}} & 1\bar{4}, \end{array}$$

where we dropped the A 's when denoting the fibers to simplify notation. Indeed we have

$$\begin{aligned} \lambda_{134} \circ \lambda_{123} &= P(1\bar{3}3\bar{4}; 1\bar{4}) \circ \mu_{(1\bar{3})(3\bar{4})} \circ P(1\bar{2}2\bar{3}; 1\bar{3}) \circ \mu_{(1\bar{2})(2\bar{3})} \\ &= P(1\bar{3}3\bar{4}; 1\bar{4}) \circ P(1\bar{2}2\bar{3}3\bar{4}; 1\bar{3}3\bar{4}) \circ \mu_{(1\bar{2}2\bar{3})3\bar{4}} \circ \mu_{(1\bar{2})(2\bar{3})} \\ &= P(1\bar{2}2\bar{3}3\bar{4}; 1\bar{4}) \circ \mu_{(1\bar{2})(2\bar{3})(3\bar{4})}. \end{aligned}$$

The second equality is true since fusion is compatible with parallel transport, and the third equality expresses the associativity of fusion. On the other

side, we have

$$\begin{aligned}
\lambda_{124} \circ \lambda_{234} &= P(1\bar{2}2\bar{4}; 1\bar{4}) \circ \mu_{(1\bar{2})(2\bar{4})} \circ P(2\bar{3}3\bar{4}; 2\bar{4}) \circ \mu_{(2\bar{3})(3\bar{4})} \\
&= P(1\bar{2}2\bar{4}; 1\bar{4}) \circ P(1\bar{2}2\bar{3}3\bar{4}; 1\bar{2}2\bar{4}) \circ \mu_{(1\bar{2})(2\bar{3}3\bar{4})} \circ \mu_{(2\bar{3})(3\bar{4})} \\
&= P(1\bar{2}2\bar{3}3\bar{4}; 1\bar{4}) \circ \mu_{(1\bar{2})(2\bar{3})(3\bar{4})}.
\end{aligned}$$

The second thing to check is that out of a connection on a Frobenius bundle over LX we obtain a superficial connection, i.e. a notion of parallel transport P along paths in LX so that any two paths in LX that are rank-two-homotopic give rise to the same parallel transport (two paths Γ and Γ' are *rank-two homotopic* if there is a homotopy $H : I \times I \rightarrow LX$ connecting Γ and Γ' so that the adjoint map $\tilde{H} : I \times I \times S^1 \rightarrow X$ has rank two, cf. [17]). If Γ' is obtained from Γ by a precomposition with a diffeomorphism of the cylinder, then $P(\Gamma') = P(\Gamma)$, by the definition of a connection. If this is not the case, they must be related by precomposition with a surjective smooth map $\Phi : I \times S^1 \rightarrow I \times S^1$. In this case, the claim is that $P(\Gamma') = E(\tilde{\Gamma} \circ \Phi)$ can be written as a composition of maps $E(\tilde{\Gamma} \circ \Phi_i)$, for some diffeomorphisms Φ_i , interspersed with parallel transport maps along paths that retrace loops, along with their inverses. Therefore, diffeomorphic-invariant parallel transport implies rank-two homotopic parallel transport. The argument sketched here is probably more transparent in the one-dimensional case of parallel transport along usual paths in a space.

Now, Waldorf [17] shows that a fusion bundle with superficial connection gives rise to an S^1 -bundle gerbe with connection over X . Conversely, an S^1 -bundle gerbe with connection over X gives rise to a rank-one 2-TFT over X by the work of Gawedzki, see [6] and [7]. We can conclude that there is an equivalence between rank-one 2-TFTs over X and S^1 -bundle gerbes with connections over X .

REFERENCES

- [1] Lowell Abrams. Two-dimensional topological quantum field theories and Frobenius algebras. *J. Knot Theory Ramifications*, 5(5):569–587, 1996.
- [2] Michael Atiyah. Topological quantum field theories. *Inst. Hautes Études Sci. Publ. Math.*, (68):175–186 (1989), 1988.
- [3] John C. Baez and James Dolan. Higher-dimensional algebra and topological quantum field theory. *J. Math. Phys.*, 36(11):6073–6105, 1995.
- [4] Kevin Costello. Topological conformal field theories and Calabi-Yau categories. *Adv. Math.*, 210(1):165–214, 2007.
- [5] Florin Dumitrescu, Stephan Stolz, and Peter Teichner. On 1-dimensional topological field theories. In preparation, 2010.
- [6] K. Gawędzki. Topological actions in two-dimensional quantum field theories. In *Non-perturbative quantum field theory (Cargèse, 1987)*, volume 185 of *NATO Adv. Sci. Inst. Ser. B Phys.*, pages 101–141. Plenum, New York, 1988.
- [7] Krzysztof Gawędzki and Nuno Reis. WZW branes and gerbes. *Rev. Math. Phys.*, 14(12):1281–1334, 2002.
- [8] Joachim Kock. *Frobenius algebras and 2D topological quantum field theories*, volume 59 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2004.

- [9] Aaron D. Lauda and Hendryk Pfeiffer. Open-closed TQFTS extend Khovanov homology from links to tangles. *J. Knot Theory Ramifications*, 18(1):87–150, 2009.
- [10] Jacob Lurie. On the classification of topological field theories. In *Current developments in mathematics, 2008*, pages 129–280. Int. Press, Somerville, MA, 2009.
- [11] Greg Moore and Graeme Segal. D-branes and k-theory in 2d topological field theory. <http://arxiv.org/abs/hep-th/0609042v1>. 2006.
- [12] Chris Schommer-Pries. The classification of two-dimensional extended topological field theories. *available online*. 2009.
- [13] Graeme Segal. Elliptic cohomology (after Landweber-Stong, Ochanine, Witten, and others). *Astérisque*, (161-162):Exp. No. 695, 4, 187–201 (1989), 1988. Séminaire Bourbaki, Vol. 1987/88.
- [14] Graeme Segal. The definition of conformal field theory. In *Topology, geometry and quantum field theory*, volume 308 of *London Math. Soc. Lecture Note Ser.*, pages 421–577. Cambridge Univ. Press, Cambridge, 2004.
- [15] Stephan Stolz and Peter Teichner. Super symmetric field theories and integral modular forms. 2007.
- [16] Stephan Stolz and Peter Teichner. What is an elliptic object? In *Topology, geometry and quantum field theory*, volume 308 of *London Math. Soc. Lecture Note Ser.*, pages 247–343. Cambridge Univ. Press, Cambridge, 2004.
- [17] Konrad Waldorf. Transgression to loop spaces and its inverse, II: Gerbes and fusion bundles with connection. <http://arxiv.org/abs/1004.0031>. 2010.
- [18] Edward Witten. The index of the Dirac operator in loop space. In *Elliptic curves and modular forms in algebraic topology (Princeton, NJ, 1986)*, volume 1326 of *Lecture Notes in Math.*, pages 161–181. Springer, Berlin, 1988.

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